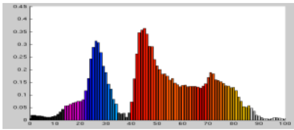
Dr. Qadri Hamarsheh



Signals and Systems

Lecture 16: Fourier Transform Analysis of Continuous Time Signals -Introduction

Outline

- Introduction.
- Existence of Fourier Transform - Dirichlet Conditions.
- Fourier Spectra.
- Examples

Introduction

- **Non-periodic** signals can be represented with the help of **Fourier Transform**.
- For **Non-periodic signals** $T_0 \to \infty$. Hence $\omega_0 \to 0$ .Therefore spacing between the spectral components becomes infinitesimal and hence the spectrum appears to be **continuous**.
- $x(t)$ - Time domain signal, $X(\omega)$ or $X(f)$ – frequency domain representation of the signal.
- The Fourier Transform Analysis of $x(t)$ is defined as:

$$\left\{\begin{array}{c} X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}\,dt \\ \boldsymbol{Or} \\ X(f) = \int_{-\infty}^{\infty} x(t)\,.\,e^{-j2\pi f t} dt \end{array}\right\} \Rightarrow\Rightarrow \textit{Forward Fourier Transform}$$

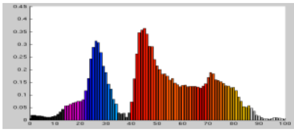
- Sometimes $X(\omega)$ is also written as $X(j\omega)$.
- Similarly $x(t)$ can be obtained from $X(\omega)$ or $X(f)$ by:

$$\left\{\begin{array}{c} x(t) = \frac{1}{2\pi}.\int_{-\infty}^{\infty} X(\omega).e^{j\omega t}\,d\omega \\ \boldsymbol{Or} \\ x(t) = \int_{-\infty}^{\infty} x(t)\,.\,e^{j2\pi f t} df \end{array}\right\} \Rightarrow\Rightarrow \textit{Inverse Fourier Transform}$$

- A **Fourier transform pair** is represented as:

$$x(t) \overset{FT}{\leftrightarrow} X(\omega) \text{ or } x(t) \overset{FT}{\leftrightarrow} X(f)$$
$$X(j\omega) = F\,[x(t)]$$
$$x(t) = F^{-1}\,[X(j\omega)]$$

- The time signal $x(t)$ is denoted by **lower case** and the frequency signal $X(\omega)$ is **capital letter**.



Existence of Fourier Transform - Dirichlet Conditions

- **As in the case of CT periodic signals, the following conditions are sufficient for the convergence of $X(\omega)$.**
  1) **Single value property.**
  2) **Finite discontinuities.**
  3) **Finite peaks (finite number of maxima and minima)**
  4) **$x(t)$ is absolutely integrable or square integrable:**

  $$\int_{-\infty}^{\infty} |x(t)|dt < \infty \rightarrow or \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

  ✓ **These conditions are sufficient, but not necessary for the signal to be Fourier Transforable.**

Fourier Spectra

- **The Fourier Transform $X(j\omega)$ of $x(t)$ is in general complex and can be expressed as**

  $$X(\omega) = |X(\omega)| < X(\omega)$$

  ✓ **The plot of $|X(\omega)|$ versus ω is called magnitude spectrum of $X(\omega)$, and the plot of $< X(\omega)$ versus ω is called phase spectrum.**

  ✓ **The amplitude (magnitude) and phase spectra are together called Fourier spectrum or frequency response of $X(\omega)$ for the frequency range $-\infty < \omega < \infty$.**

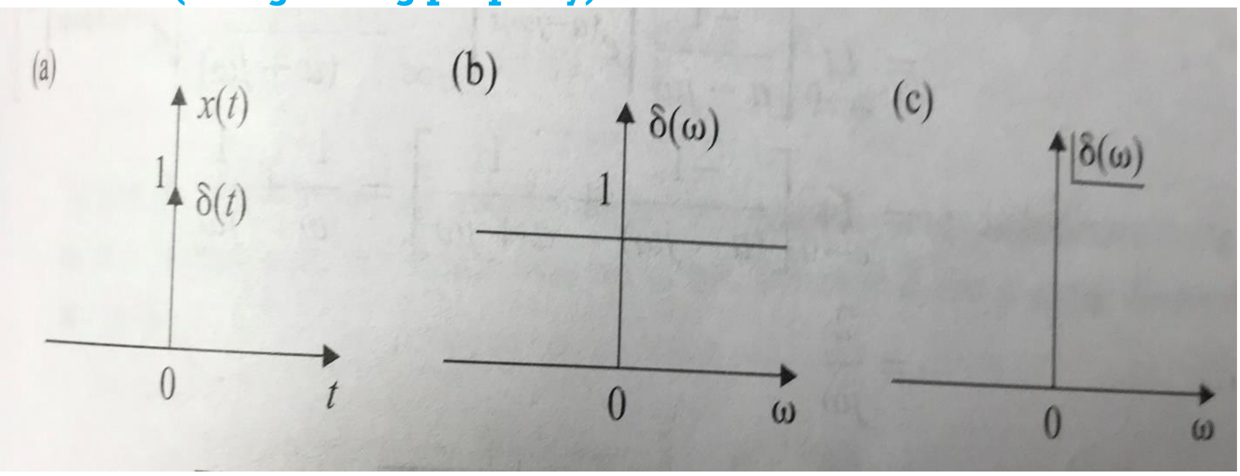
Examples

Find the Fourier transform of the following time signals and sketch their Fourier Spectra (amplitude and phase).

1) $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt = e^{-j\omega.0} = 1$$

(Using sifting property)



Representation of $\delta(t)$and its spectra

2) $x(t) = e^{-at}.u(t),\ a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-at}.u(t)e^{-j\omega t}\,dt$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at}.e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

Thus

$$e^{-at}u(t) \overset{FT}{\leftrightarrow} \frac{1}{a+j\omega}$$

To obtain magnitude and phase Spectrum:

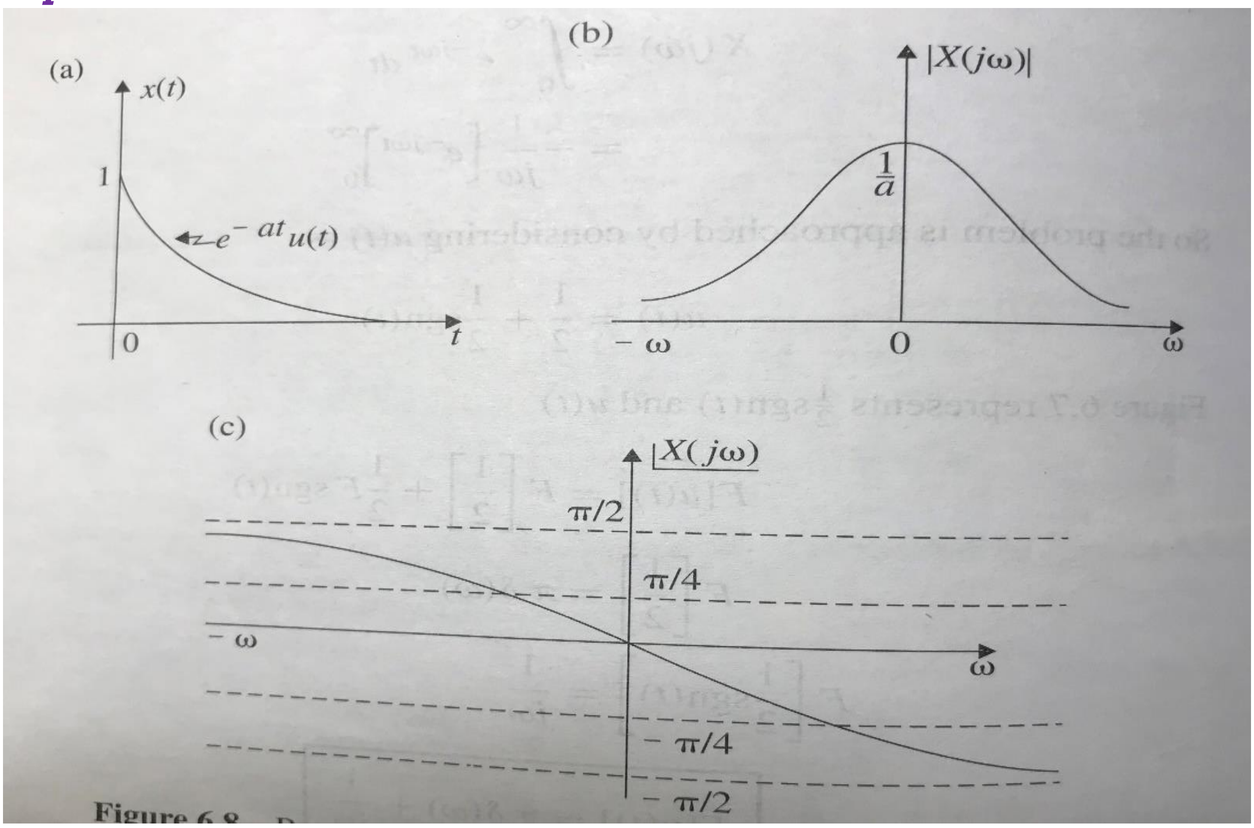
$$X(\omega) = \frac{1}{a+j\omega} = \frac{1}{a+j\omega} \times \frac{a-j\omega}{a-j\omega} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

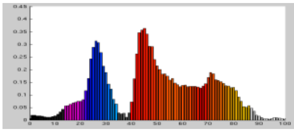
$$|X(\omega)| = \sqrt{\left(\frac{a}{a^2+\omega^2}\right)^2 + \left(\frac{\omega}{a^2+\omega^2}\right)^2} = \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}}$$

$$|X(\omega)| = \sqrt{\frac{1}{a^2+\omega^2}}$$

$$< X(\omega) = tan^{-1}\left[\frac{\frac{-\omega}{a^2+\omega^2}}{\frac{a}{a^2+\omega^2}}\right] = -tan^{-1}\left(\frac{\omega}{a}\right)$$

The following figures show the representation of $x(t) = e^{-at}.u(t)$ ***and its FT spectra.***



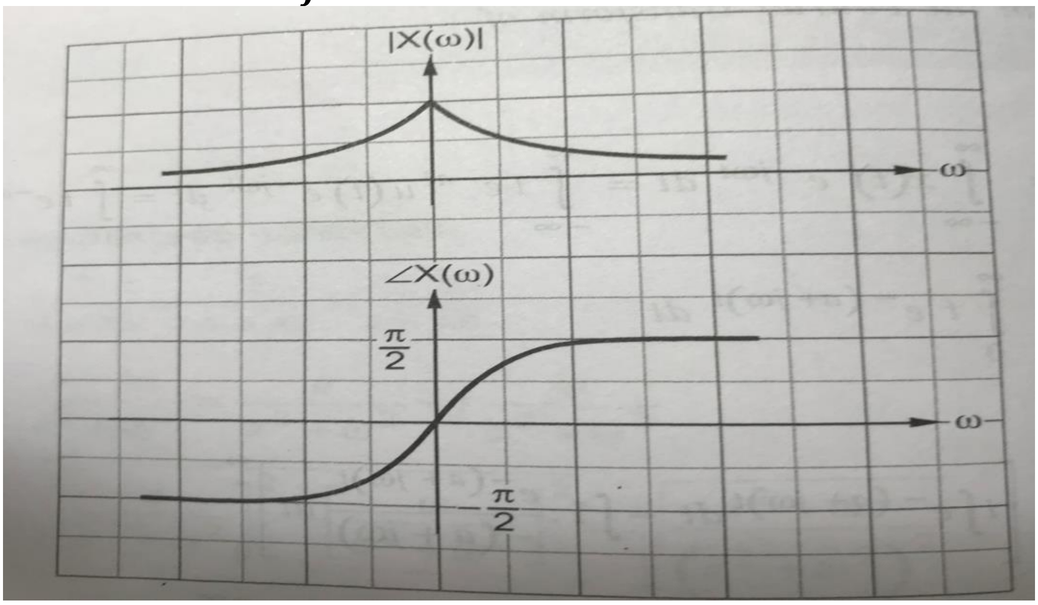


3) $x(t) = e^{at}.u(-t), \;\; a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{at}.u(-t)e^{-j\omega t}\,dt$$

$$X(j\omega) = \int_{-\infty}^{0} e^{at}.e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt = \frac{e^{(a-j\omega)t}}{a-j\omega}\Bigg|_{-\infty}^{0} = \frac{1}{a-j\omega}$$

$$e^{at}.u(-t) \overset{FT}{\leftrightarrow} \frac{1}{a-j\omega}$$



4) $x(t) = e^{at}.u(t), \;\; a > 0$

$$X(j\omega) = \int_{0}^{\infty} e^{at}.e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{(a-j\omega)t}\,dt$$

$$= \frac{1}{a-j\omega}\left[e^{(a-j\omega)t}\right]_{0}^{\infty}$$

If the upper limit is applied to the above integral, the Fourier integral does not converge. Hence FT does not exist for $x(t) = e^{at}.u(t)$.

5) $x(t) = e^{-a|t|}, a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{at}.e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}.e^{-j\omega t}\,dt$$

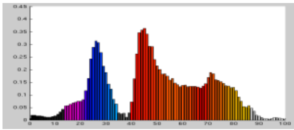
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{1}{a-j\omega}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} - \frac{1}{(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \Longrightarrow$$

$$X(j\omega) = \frac{2a}{a^2 - \omega^2}$$

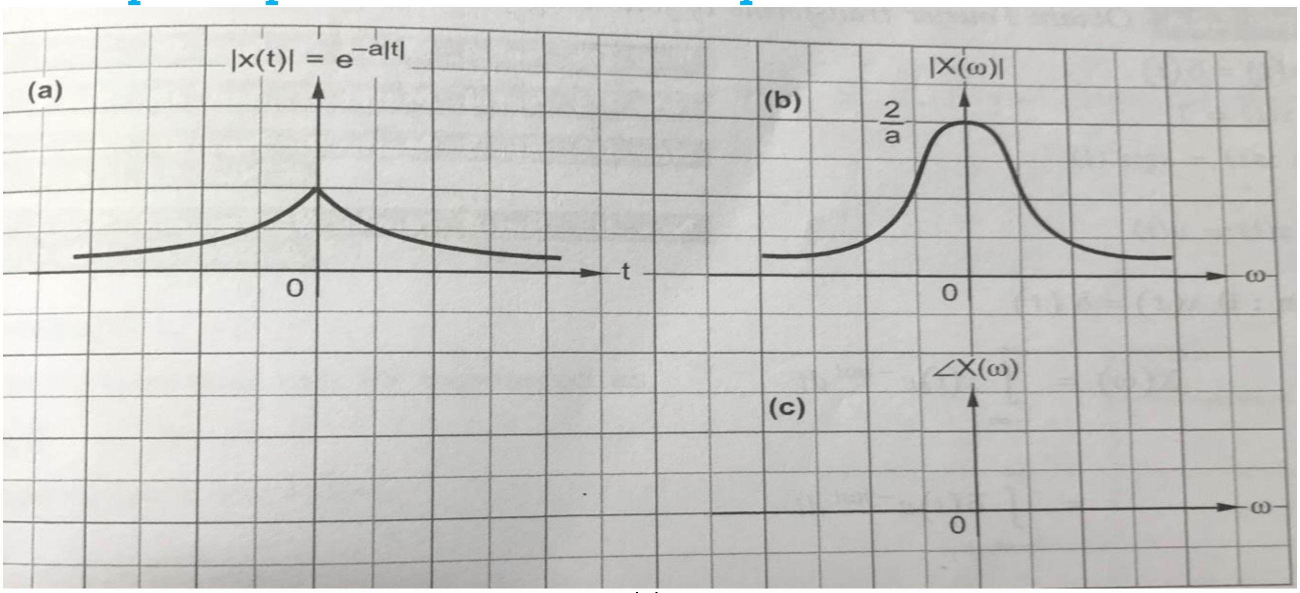
$$e^{-a|t|} \overset{FT}{\leftrightarrow} \frac{2a}{a^2+\omega^2}$$



Fourier Spectra:

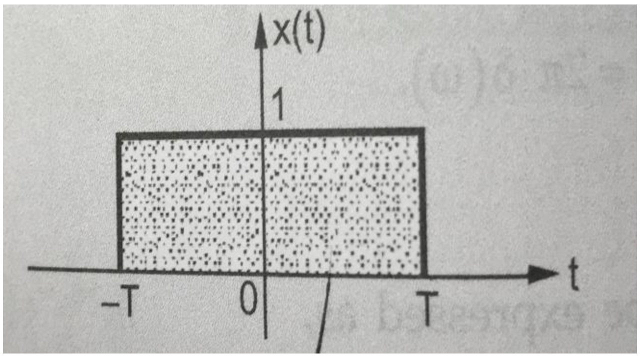
$$|X(j\omega)| = \frac{2a}{a^2+\omega^2}\,, \qquad < X(\omega) = 0$$

The phase spectrum is zero for all frequencies:



Representation of $x(t) = e^{-a|t|}$ and its frequency spectrum

6) Obtain the Fourier transform of a rectangular pulse as shown in the following figure:



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-T}^{T} 1.\,e^{-j\omega t}\,dt$$

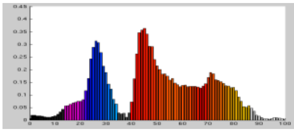
$$= \left[-\frac{e^{-j\omega t}}{j\omega}\right]_{-T}^{T} = \frac{-1}{j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right]$$

$$= \frac{2}{\omega}.\left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right] = \frac{2}{\omega}\,.\sin(\omega T)$$

We know that

$$sinc(\theta) = \frac{\sin(\pi\theta)}{\pi\,\theta}$$

Hence rearranging above equation,



$$X(\omega) = 2T\frac{\sin(\pi.\frac{\omega T}{\pi})}{\pi.\frac{\omega T}{\pi}} = 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$$

Thus:

$$(\text{Rectangular pulse amplitude } 1, \quad \text{period } 2T) \overset{FT}{\leftrightarrow} 2T\,\mathrm{sinc}\left(\frac{\omega t}{\pi}\right)$$

- ✓ **A Rectangular pulse amplitude A and width 2T is represented by 'rect' Function:**

$$Rect\left(\frac{t}{(2T)}\right) = \begin{cases} A & for\ -T \le t \le T \\ 0 & elsewhere \end{cases}$$

Hence for the rectangular pulse

$$Rect\left(\frac{t}{(2t)}\right) \overset{FT}{\leftrightarrow} 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$$

- ✓ **Magnitude and phase plot**

Since $X(\omega)$ is real

$$\Rightarrow |X(j\omega)| = 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$$

And

$$< X(\omega) = 0$$

We know that

$$\Rightarrow |X(\omega)| = 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right) = \frac{2}{\omega}.\sin(\omega T)$$

Note that this function goes to zero at

$$\omega = \pm\frac{\pi}{T}, \pm\frac{2\pi}{T}, \pm\frac{3\pi}{T}, \dots\dots\dots$$

By L' Hopitals rule:

$$\lim_{\omega \to 0} \frac{2}{\omega}.\sin(\omega T) = 2T, \qquad Hence\ X(\omega) = 2T \quad at \quad \omega = 0$$